1 Finite-Difference Methods for the One Dimensional Wave Equation

A one-dimensional for of the wave equation can be found as follows,

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2(x)\frac{\partial^2 u}{\partial x^2} + f(x,t), & (x,t) \in (a,b)\times(T_1,T_2], \\ u(x,T_1) = h(x), & x \in [a,b], \\ \frac{\partial}{\partial t}u(x,T_1) = s(x), & x \in [a,b], \\ u(a,t) = f_1(t), & t \in [T_1,T_2], \\ u(b,t) = f_2(t), & t \in [T_1,T_2]. \end{cases} \tag{1}$$

Moreover, we derived this a higher-order scheme as follows,

$$\left(1 + \frac{1^2}{12}\delta_t^2\right)^{-1}\frac{\delta_t^2}{\tau^2}u_i^n = c_i^2\left(1 + \frac{1}{12}\delta_x^2\right)^{-1}\frac{\delta_x^2}{h^2}u_i^n + f_i^n, \tag{2}$$

Theorem 1 (Stability)**.** *Assume that the solution of the two-dimensional acoustic wave equation* (1) *is sufficiently smooth. The stability criteria of the numerical scheme given in* (2) *can be derived as follows,*

$$\lambda \max_{x\in\Omega}\{|c(x)|\} = c\lambda < 1. \tag{3}$$

Proof. First off, without loss of generality, in the equation (2), assume that $f_i^n = 0$ for all values of i and n. Moreover, let $c = \max_{(x)\in\Omega}\{|c(x)|\}$. It follows that,

$$\left(1 + \frac{1-c^2\lambda^2}{12}\delta_x^2\right)\delta_t^2 u_{m,j}^n = c^2\lambda^2\delta_x^2 u_{m,j}^n. \tag{4}$$

Considering $\rho = \frac{1-c^2\lambda^2}{12}$, it follows from (4)

$$\left(1+\rho\delta_x^2\right)\left[u_{i,j}^{n+1} + u_{i,j}^{n-1}\right] - \left[2\left(1+\rho\delta_x^2\right) + c^2\lambda^2\delta_x^2\right]u_{m,j}^n = 0. \tag{5}$$

Applying discrete Fourier transformation on (5) leads to the following result,

$$\left(1+\rho\delta_x^2\right)e^{ih(m\xi)}\left[\hat{u}^{n+1} + \hat{u}^{n-1}\right] - \left[2\left(1+\rho\delta_x^2\right) + c^2\lambda^2\delta_x^2\right]e^{ih(m\xi)}\hat{u}^n = 0, \tag{6}$$

which also can be expressed as follows, considering $\theta = h\xi$,

$$\left(1+\rho\delta_x^2\right)e^{im\theta}\left[\hat{u}^{n+1} + \hat{u}^{n-1}\right] - \left[2\left(1+\rho\delta_x^2\right)e^{im\theta} + c^2\lambda^2\delta_x^2 e^{im\theta}\right]\hat{u}^n = 0. \tag{7}$$

It follows that,

$$\begin{aligned} e^{im\theta}\left[\rho\left(e^{\theta}+e^{-\theta}\right) + (1-2\rho)\right]\left[\hat{u}^{n+1} + \hat{u}^{n-1}\right] - \Big[2e^{im\theta}\left[\rho\left(e^{\theta}+e^{-\theta}\right) + (1-2\rho)\right] \\ + c^2\lambda^2 e^{im\theta}\left[2\left(e^{\theta}+e^{-\theta}\right) - 2\right]\Big]\hat{u}^n = 0. \end{aligned}$$

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This can be simplified as follows,

$$e^{im\theta}\left(2\rho\cos\theta + (1-2\rho)\right)\left[\hat{u}^{n+1} + \hat{u}^{n-1}\right] - \left[2e^{im\theta}\left(2\rho\cos\theta + (1-2\rho)\right) + c^2\lambda^2 e^{im\theta}\left(2\cos\theta - 2\right)\right]\hat{u}^n = 0. \tag{8}$$

Dividing by $e^{im\theta}$,

$$\left(1 - 4\rho\sin^2\frac{\theta}{2}\right)\left[\hat{u}^{n+1} + \hat{u}^{n-1}\right] - \left[2\left(1 - 4\rho\sin^2\frac{\theta}{2}\right) - 4c^2\lambda^2\sin^2\frac{\theta}{2}\right]\hat{u}^n = 0, \tag{9}$$

It follow from the fact that $\rho = \frac{1-c^2\lambda^2}{12}$,

$$1 - 4\rho\sin^2\frac{\theta}{2} = 1 - \frac{1}{3}\sin^2\frac{\theta}{2} + \frac{c^2\lambda^2}{3}\sin^2\frac{\theta}{2} > 0$$

Furthermore, set $S = \sin\frac{\theta}{2}$. It follows from (8) that,

$$\left(1-4\rho S^2\right)\left[\hat{u}^{n+1}+\hat{u}^{n-1}\right]-\left[2\left(1-4\rho S^2\right)-4c^2\lambda^2 S^2\right]\hat{u}^n = 0. \tag{10}$$

Hence,

$$\hat{u}^{n+1}+\left(4c^2\lambda^2\frac{S^2}{(1-4\rho S^2)}-2\right)\hat{u}^n+\hat{u}^{n-1} = 0. \tag{11}$$

As a result,

$$\hat{u}^{n+1}-2\varphi(\theta)\hat{u}^n+\hat{u}^{n-1} = 0, \tag{12}$$

where, $\varphi(\theta) = 1-2c^2\lambda^2\frac{S^2}{\left(1-4\rho S^2\right)}$. It follows that,

$$g^2-2\varphi(\theta)g+1 = 0, \tag{13}$$

where g is the amplification factor.

In case that, $|\varphi(\theta)| \geq 1$ then quadratic equation (13) has two real roots and one of which is bigger than 1. This would lead to instability. Therefore, The necessary and sufficient condition for the scheme to be stable can be considered as follows,

$$|\varphi(\theta)| < 1$$

On one hand, clearly, $S^2 = \sin^2\frac{\theta}{2} \leq 1$. Hence, $\varphi(\theta,\theta_y) < 1$.

On the other hand,

$$1-2c^2\lambda^2\frac{S^2}{(1-4\rho S^2)} > -1. \tag{14}$$

$$c^2\lambda^2\frac{S^2}{(1-4\rho S^2)} < 1. \tag{15}$$

Define,

$$\chi(S^2) = \frac{S^2}{(1-4\rho S^2)} \tag{16}$$

If $c^2\lambda^2 = 1$, $c^2\lambda^2\chi(1) = 1$, the inequality in the equation (15) is not satisfied, the method is not stable. Therefore, $c^2\lambda^2 < 1$ should be considered. It is easy to see that the numerator of $\chi(S^2)$ is an increasing function, whereas the denominator of is a decreasing. It follows that, $\chi(S^2)$ takes its maximum at (1). Hence,

$$c^2\lambda^2\chi(1) < 1. \tag{17}$$

This means,

$$\frac{c^2\lambda^2}{\left(\frac{2}{3}+\frac{1}{3}c^2\lambda^2\right)} < 1. \tag{18}$$

Therefore, the scheme is stable if

$$\lambda\max_{x\in\Omega}\{|c(x)|\} = c\lambda < 1. \tag{19}$$

□

Theorem 2 (Convergence)**.** *The numerical scheme* (2) *is fourth-order in both time and space considering* $h = h_x = h_y$.

Proof. Without loss of generality, let's assume $f(x,t) = 0$ for all x and t. Moreover, let $c = \max_{(x)\in\Omega}\{|c(x)|\}$.

As a result, the equation (2) reduces to the following form,

$$\left(1+\frac{1}{12}\delta_x^2\right)\frac{\delta_t^2}{\tau^2}\frac{u_i^n}{c^2} = \left(1+\frac{1}{12}\delta_t^2\right)\frac{\delta_x^2}{h^2}u_i^n. \tag{20}$$

We know,

$$\left(1+\frac{1}{12}\delta_x^2\right)\frac{\delta_t^2}{\tau^2}\frac{u_i^n}{c^2} = \frac{d^2}{dt^2}\frac{u_i^n}{c^2} + \frac{1}{12}\frac{d^4}{dt^4}\frac{u_i^n}{c^2}\tau^2 + \frac{1}{12}\frac{d^4}{dx^2dt^2}\frac{u_i^n}{c^2}h^2 + \frac{1}{144}\frac{d^6}{dx^2dt^4}\frac{u_i^n}{c^2}h^2\tau^2 + \frac{1}{360}\frac{d^6}{dt^6}\frac{u_i^n}{c^2}\tau^4 + \frac{1}{144}\frac{d^6}{dx^4dt^2}\frac{u_i^n}{c^2}h^4 + \mathscr{O}(h^6+\tau^6). \tag{21}$$

and

$$\left(1+\frac{1}{12}\delta_t^2\right)\frac{\delta_x^2}{h^2}u_i^n = \frac{d^2}{dx^2}u_i^n + \frac{1}{12}\frac{d^4}{dx^4}u_i^n h^2 + \frac{1}{12}\frac{d^4}{dt^2dx^2}u_i^n\tau^2 + \frac{1}{144}\frac{d^6}{dt^2dx^4}u_i^n h^2\tau^2 + \frac{1}{360}\frac{d^6}{dx^6}u_i^n h^4 + \frac{1}{144}\frac{d^6}{dt^4dx^2}u_i^n\tau^4 + \mathscr{O}(h^6+\tau^6) \tag{22}$$

Therefore,

$$\left(1+\frac{1}{12}\delta_x^2\right)\frac{\delta_t^2}{\tau^2}\frac{u_i^n}{c^2} - \left(1+\frac{1}{12}\delta_t^2\right)\frac{\delta_x^2}{h^2}u_i^n = \frac{d^2}{dt^2}\frac{u_i^n}{c^2} - \frac{d^2}{dx^2}u_i^n + \frac{1}{144}\left(\frac{d^4}{dx^2dt^4}\frac{u_i^n}{c^2} - \frac{d^4}{dt^2dx^4}u_i^n\right)h^2\tau^2 + \frac{1}{144}\frac{d^6}{dx^4dt^2}\frac{u_i^n}{c^2}h^4 - \frac{1}{360}\frac{d^6}{dx^6}u_i^n h^4 + \frac{1}{360}\frac{d^6}{dt^6}\frac{u_i^n}{c^2}\tau^4 - \frac{1}{144}\frac{d^6}{dt^4dx^2}u_i^n\tau^4 + \mathscr{O}(h^6+\tau^6) \tag{23}$$

The order of method respect to h can be easily obtained if $\tau \to 0$. In other words,

$$\left(1+\frac{1}{12}\delta_x^2\right)\frac{\delta_t^2}{\tau^2}\frac{u_i^n}{c^2} - \left(1+\frac{1}{12}\delta_t^2\right)\frac{\delta_x^2}{h^2}u_i^n = \frac{d^2}{dt^2}\frac{u_i^n}{c^2} - \frac{d^2}{dx^2}u_i^n + \frac{1}{144}\frac{d^6}{dx^4dt^2}\frac{u_i^n}{c^2}h^4 - \frac{1}{360}\frac{d^6}{dx^6}u_i^n h^4 + \mathscr{O}(h^6) \tag{24}$$

The following relationship between h and τ is considered to find the order of method to τ,

$$h = \mu\tau \tag{25}$$

where μ is a known integer.

Therefore,

$$\left(1+\frac{1}{12}\delta_x^2\right)\frac{\delta_t^2}{\tau^2}\frac{u_i^n}{c^2} - \left(1+\frac{1}{12}\delta_t^2\right)\frac{\delta_x^2}{h^2}u_i^n = \frac{1}{144}\left(\frac{d^4}{dx^2dt^4}\frac{u_i^n}{c^2} - \frac{d^4}{dt^2dx^4}u_i^n\right)\mu^2\tau^4 + \frac{1}{144}\frac{d^6}{dx^4dt^2}\frac{u_i^n}{c^2}\mu^4\tau^4 - \frac{1}{360}\frac{d^6}{dx^6}u_i^n\mu^4\tau^4 + \frac{1}{360}\frac{d^6}{dt^6}\frac{u_i^n}{c^2}\tau^4 - \frac{1}{144}\frac{d^6}{dt^4dx^2}u_i^n\tau^4 + \mathscr{O}(\tau^6), \tag{26}$$

which means that the scheme is 4th order respect to τ. □