The Finite-Difference Method for the three-dimensional wave equation

Let $I=(a,b)\times(c,d)\times(e,f)$ and $J=[a,b]\times[c,d]\times[e,f]$. A three-dimensional for of the wave equation can be found as follows,

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}=c^2(x,y,z)\Delta u+f(x,y,z,t), & (x,y,z)\in I,\ t\in(T_1,T_2],\\ u(x,y,z,T_1)=g(x,y,z), & (x,y,z)\in J,\\ \frac{\partial}{\partial t}u(x,y,z,T_1)=s(x,y,z), & (x,y,z)\in J,\\ u(a,y,z,t)=f_a(y,z,t), & (y,z)\in[c,d]\times[e,f],\ t\in[T_1,T_2],\\ u(b,y,z,t)=f_b(y,z,t), & (y,z)\in[c,d]\times[e,f],\ t\in[T_1,T_2],\\ u(x,c,z,t)=f_c(x,z,t), & (x,z)\in[a,b]\times[e,f],\ t\in[T_1,T_2],\\ u(x,d,z,t)=f_d(x,z,t), & (x,z)\in[a,b]\times[e,f],\ t\in[T_1,T_2],\\ u(x,y,e,t)=f_e(x,y,t), & (x,y)\in[a,b]\times[c,d],\ t\in[T_1,T_2],\\ u(x,y,f,t)=f_f(x,y,t), & (x,y)\in[a,b]\times[c,d],\ t\in[T_1,T_2].\end{cases} \tag{1}$$

Moreover, we derived the following implicit scheme in the main article,

$$\frac{1}{\tau^2}\delta_t^2 u_{i,j,k}^n=c_{i,j,k}^2\left(\frac{1}{h_x^2}\delta_x^2+\frac{1}{h_y^2}\delta_y^2+\frac{1}{h_z^2}\delta_z^2\right)\left[\alpha u_{i,j,k}^{n+1}+\beta u_{i,j,k}^n+\gamma u_{i,j,k}^{n-1}\right]+f_{i,j,k}^n, \tag{2}$$

Theorem 1 (Stability)**.** *Assume that the solution of the two dimensional acoustic wave equation* (1) *is sufficiently smooth. The stability criteria of the numerical scheme introduced in* (2) *can be derived as follows,*

$$\tau \max_{0\le m\le N_x,\ 0\le j\le N_y,\ 0\le k\le N_z} c_{m,j,k}<\frac{1}{\sqrt{(1-4\alpha)}}\frac{1}{\sqrt{3}}h, \tag{3}$$

where $\alpha=\gamma$ and $h=h_x=h_y=h_z$.

Proof. Let $\alpha=\gamma$, and $f(x,y,t)=0$. The scheme introduced in (**??**) can be written in the following form,

$$\begin{aligned}&-\alpha c_{m,j,k}^2\left[\lambda_x^2\left(u_{m+1,j,k}^{n+1}+u_{m-1,j,k}^{n+1}\right)+\lambda_y^2\left(u_{m,j-1,k}^{n+1}+u_{m,j+1,k}^{n+1}\right)+\lambda_z^2\left(u_{m,j,k-1}^{n+1}+u_{m,j,k+1}^{n+1}\right)\right]\\ &+\left[1+2\alpha c_{m,j,k}^2\left(\lambda_x^2+\lambda_y^2+\lambda_z^2\right)\right]u_{m,j,k}^{n+1}=\beta c_{m,j,k}^2\left[\lambda_x^2\left(u_{m+1,j,k}^n+u_{m-1,j,k}^n\right)\right.\\ &\left.+\lambda_y^2\left(u_{m,j-1,k}^n+u_{m,j+1,k}^n\right)+\lambda_z^2\left(u_{m,j,k-1}^n+u_{m,j,k+1}^n\right)\right]+2\left[1-\beta c_{m,j,k}^2\left(\lambda_x^2+\lambda_y^2+\lambda_z^2\right)\right]u_{m,j,k}^n\\ &+\alpha c_{m,j,k}^2\left[\lambda_x^2\left(u_{m+1,j,k}^{n-1}+u_{m-1,j,k}^{n-1}\right)+\lambda_y^2\left(u_{m,j-1,k}^{n-1}+u_{m,j+1,k}^{n-1}\right)+\lambda_z^2\left(u_{m,j,k-1}^{n-1}+u_{m,j,k+1}^{n-1}\right)\right]\\ &-\left[1+2\alpha c_{m,j,k}^2\left(\lambda_x^2+\lambda_y^2+\lambda_z^2\right)\right]u_{m,j,k}^{n-1}.\end{aligned} \tag{4}$$

It can be concluded that,

$$\begin{aligned}&\left[-\alpha c_{m,j,k}^2\left[\lambda_x^2\left(e^{-ih_x\xi_x}+e^{ih_x\xi_x}\right)+\lambda_y^2\left(e^{-ih_y\xi_y}+e^{ih_y\xi_y}\right)+\lambda_z^2\left(e^{-ih_z\xi_z}+e^{ih_z\xi_z}\right)\right]\right.\\ &\left.+\left[1+2\alpha c_{m,j,k}^2\left(\lambda_x^2+\lambda_y^2+\lambda_z^2\right)\right]\right]\left(\hat{u}^{n+1}(\xi_x,\xi_y,\xi_z)+\hat{u}^{n-1}(\xi_x,\xi_y,\xi_z)\right)\\ &-\left[\beta c_{m,j,k}^2\left[\lambda_x^2\left(e^{-ih_x\xi_x}+e^{ih_x\xi_x}\right)+\lambda_y^2\left(e^{-ih_y\xi_y}+e^{ih_y\xi_y}\right)+\lambda_z^2\left(e^{-ih_z\xi_z}+e^{ih_z\xi_z}\right)\right]\right.\\ &\left.+2\left[1-\beta c_{m,j,k}^2\left(\lambda_x^2+\lambda_y^2+\lambda_z^2\right)\right]\right]\hat{u}^n(\xi_x,\xi_y,\xi_z).\end{aligned} \tag{5}$$

It follows that,

$$\begin{aligned}&\left[-\alpha c_{m,j,k}^2\left[\lambda_x^2\left(e^{-i\theta_x}+e^{i\theta_x}\right)+\lambda_y^2\left(e^{-i\theta_y}+e^{i\theta_y}\right)+\lambda_z^2\left(e^{-i\theta_z}+e^{i\theta_z}\right)\right]\right.\\ &\left.+\left[1+2\alpha c_{m,j,k}^2\left(\lambda_x^2+\lambda_y^2+\lambda_z^2\right)\right]\right]\left(g+g^{-1}\right)\\ &-\left[\beta c_{m,j,k}^2\left[\lambda_x^2\left(e^{-i\theta_x}+e^{i\theta_x}\right)+\lambda_y^2\left(e^{-i\theta_y}+e^{i\theta_y}\right)+\lambda_z^2\left(e^{-i\theta_z}+e^{i\theta_z}\right)\right]\right.\\ &\left.+2\left[1-\beta c_{m,j,k}^2\left(\lambda_x^2+\lambda_y^2+\lambda_z^2\right)\right]\right],\end{aligned} \tag{6}$$

where $\theta_x = h_x\xi_x$, $\theta_y = h_y\xi_y$, $\theta_z = h_z\xi_z$ and g is the amplification factor. The equation (6) can be simplified as follows,

$$g^2 + 2\left(\frac{1-2(1-2\alpha)\varphi}{1+4\alpha\varphi}\right)g + 1 = 0, \tag{7}$$

where $\varphi = c_{m,j,k}^2\left[\lambda_x^2\sin^2\frac{\theta_x}{2} + \lambda_y^2\sin^2\frac{\theta_y}{2} + \lambda_z^2\sin^2\frac{\theta_z}{2}\right]$.

In case that $\left(\frac{1-2(1-2\alpha)\varphi}{1+4\alpha\varphi}\right)^2 - 1 \geq 0$, the quadratic equation (7) will have two distinct real roots and one of them is greater than 1. This would lead to instability.

Therefore, the following case is considered in which the roots of quadratic equation (7) are complex.

$$\left(\frac{1-2(1-2\alpha)\varphi}{1+4\alpha\varphi}\right)^2 - 1 < 0. \tag{8}$$

As a result,

$$-16\alpha\varphi^2 + 4\varphi^2 - 4\varphi < 0. \tag{9}$$

Since φ is non-zero,

$$(1-4\alpha)\varphi < 1. \tag{10}$$

This means,

$$(1-4\alpha)c_{m,j,k}^2\left[\lambda_x^2\sin^2\frac{\theta_x}{2} + \lambda_y^2\sin^2\frac{\theta_y}{2} + \lambda_z^2\sin^2\frac{\theta_z}{2}\right] < 1, \tag{11}$$

for any θ_x, θ_y and θ_z. Therefore,

$$\tau \max_{0\leq m\leq N_x,\ 0\leq j\leq N_y,\ 0\leq k\leq N_z} c_{m,j,k} < \frac{1}{\sqrt{(1-4\alpha)}}\frac{1}{\sqrt{3}}h, \tag{12}$$

where $h = h_x = h_y = h_z$. □

Corollary 1.1. *The following results can be obtained from theorem 1*

I. The stability analysis criteria of the explicit scheme, introduced in (**??**)*, can be obtained by setting* $\alpha = 0$ *in the equation* (3)

$$\tau \max_{0\leq m\leq N_x,\ 0\leq j\leq N_y,\ 0\leq k\leq N_z} c_{m,j,k} < \frac{1}{\sqrt{3}}h. \tag{13}$$

II. Similarly, the stability criteria of the implicit scheme can be derived by setting $\alpha = 1/4$ *in* (**??**)*. It follows that,*

$$\tau \max_{0\leq m\leq N_x,\ 0\leq j\leq N_y,\ 0\leq k\leq N_z} c_{m,j,k} < \infty, \tag{14}$$

which means that the implicit scheme is unconditionally stable.

Theorem 2 (Convergence)**.** *The numerical scheme* (**??**) *is the second order in both time and space considering* $h = h_x = h_y = h_z$*.*

Proof. Let's assume $h = h_x = h_y = h_z$ and also assume $f_{i,j,k}^n = 0$ in the equation (2) for all values of i, j and n. It follows that,

$$\frac{1}{\tau^2}\delta_t^2 u_{i,j,k}^n = \frac{c_{i,j,k}^2}{h^2}\Delta\left[\alpha u_{i,j,k}^{n+1} + \beta u_{i,j,k}^n + \gamma u_{i,j,k}^{n-1}\right]. \tag{15}$$

The difference between the left hand and right sides of (15), using Taylor series expansion, can be found as follows,

$$
\begin{aligned}
&\frac{1}{\tau^2}\delta_t^2 u^n_{i,j,k} - \frac{c^2_{i,j,k}}{h^2}\Delta\left[\alpha u^{n+1}_{i,j,k} + \beta u^n_{i,j,k} + \gamma u^{n-1}_{i,j,k}\right] = \frac{d^2}{dt^2}u^n_{i,j,k} - c^2_{i,j,k}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\Big(\alpha u^{n+1}_{i,j,k} \\
&+\beta u^n_{i,j,k} + \gamma u^n_{i,j,k}\Big) + \frac{1}{12}\left[\tau^2\frac{d^4}{dt^4}u^n_{i,j,k} - c^2_{i,j,k}h^2\left(\frac{d^4}{dx^4} + \frac{d^4}{dy^4} + \frac{d^4}{dz^4}\right)\left(\alpha u^{n+1}_{i,j,k} + \beta u^n_{i,j,k} + \gamma u^{n-1}_{i,j,k}\right)\right] \\
&+ \frac{1}{360}\left[\tau^4\frac{d^6}{dt^6}u^n_{i,j,k} - c^2_{i,j,k}h^4\left(\frac{d^6}{dx^6} + \frac{d^6}{dy^6} + \frac{d^6}{dz^6}\right)\left(\alpha u^{n+1}_{i,j,k} + \beta u^n_{i,j,k} + \gamma u^{n-1}_{i,j,k}\right)\right] + \mathscr{O}(h^6 + \tau^6). \qquad (16)
\end{aligned}
$$

The order of general scheme in space can be obtained by choosing τ arbitrary small in the equation (17). It follows that,

$$
\begin{aligned}
&\frac{1}{\tau^2}\delta_t^2 u^n_{i,j,k} - \frac{c^2_{i,j,k}}{h^2}\Delta\left[\alpha u^{n+1}_{i,j,k} + \beta u^n_{i,j,k} + \gamma u^{n-1}_{i,j,k}\right] = \frac{d^2}{dt^2}u^n_{i,j,k} - c^2_{i,j,k}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\Big(\alpha u^{n+1}_{i,j,k} \\
&+\beta u^n_{i,j,k} + \gamma u^n_{i,j,k}\Big) - \frac{1}{12}c^2_{i,j,k}h^2\left(\frac{d^4}{dx^4} + \frac{d^4}{dy^4} + \frac{d^4}{dz^4}\right)\left(\alpha u^{n+1}_{i,j,k} + \beta u^n_{i,j,k} + \gamma u^{n-1}_{i,j,k}\right) + \mathscr{O}(h^4). \qquad (17)
\end{aligned}
$$

The implicit scheme, $\alpha = \gamma = \frac{1}{4}$ and $\beta = \frac{1}{2}$, is unconditionally stable; therefore, the order of general scheme in time can be obtained by choosing h arbitrary small in the equation (16). It follows that,

$$
\begin{aligned}
&\frac{1}{\tau^2}\delta_t^2 u^n_{i,j,k} - \frac{c^2_{i,j,k}}{h^2}\Delta\left[\alpha u^{n+1}_{i,j,k} + \beta u^n_{i,j,k} + \gamma u^{n-1}_{i,j,k}\right] = \frac{d^2}{dt^2}u^n_{i,j,k} - c^2_{i,j,k}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\Big(\alpha u^{n+1}_{i,j,k} \\
&+\beta u^n_{i,j,k} + \gamma u^n_{i,j,k}\Big) + \frac{\tau^2}{12}\frac{d^4}{dt^4}u^n_{i,j,k} + \mathscr{O}(\tau^4). \qquad (18)
\end{aligned}
$$

However, the following relationship between h and τ can be considered if the method is not unconditionally stable,

$$
h = \mu\tau, \qquad (19)
$$

where μ is a positive real number. It follows from (20)

$$
\begin{aligned}
&\frac{1}{\tau^2}\delta_t^2 u^n_{i,j,k} - \frac{c^2_{i,j,k}}{h^2}\Delta\left[\alpha u^{n+1}_{i,j,k} + \beta u^n_{i,j,k} + \gamma u^{n-1}_{i,j,k}\right] = \frac{d^2}{dt^2}u^n_{i,j,k} - c^2_{i,j,k}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\Big(\alpha u^{n+1}_{i,j,k} \\
&+\beta u^n_{i,j,k} + \gamma u^n_{i,j,k}\Big) + \frac{\tau^2}{12}\frac{d^4}{dt^4}u^n_{i,j,k} - \mu^2\tau^2\frac{c^2_{i,j,k}}{12}\left(\frac{d^4}{dx^4} + \frac{d^4}{dy^4}\right)\left(\alpha u^{n+1}_{i,j,k} + \beta u^n_{i,j,k} + \gamma u^{n-1}_{i,j,k}\right) + \mathscr{O}(\tau^4), \qquad (20)
\end{aligned}
$$

which confirms that the conditionally stable scheme is second order in time as well. □