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A stable numerical solution of a class of semi-linear Cauchy problems

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Abstract. In this paper, we are mainly concerned with the numerical solution of a class of non-well posed semi-linear Cauchy problems for the heat equation. The noisy data are given at the boundary. A stable numerical method based on mollification scheme and marching method is developed to solve the proposed problem. The error of this method is analyzed and some numerical examples are investigated.

Keywords: Nonlinear Cauchy problem; Marching scheme; Mollification method.

Mathematics Subject Classification 2010: 35K55, 65F22, 65M12.

1 Introduction

Seeking the exact and approximate solutions of the nonlinear partial differential equations (PDE's) play an important role in the nonlinear problems which may describe different phenomenon. Many different analytical and numerical methods have been investigated to handle the nonlinear problems in literature (see [2-4], [6-8], [12, 13]).

This paper investigates a class of non-well posed semi-linear Cauchy problem for the heat equation defined by

$$u_t = u_{xx} + \varphi(u), \qquad 0 < x < 1,\ t > 0, \tag{1.1}$$
$$u(0,t) = \alpha(t), \qquad t > 0, \tag{1.2}$$
$$u_x(0,t) = \beta(t), \qquad t > 0, \tag{1.3}$$

where $\varphi(u) = \sum_{i=0}^{n} a_i u^i$ shows the polynomial nonlinear term and the coefficients $a_i;\ i = 0, 1, \cdots, n$ are considered to be known parameters.

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The nonlinear problem (1.1)-(1.3) is a well-known mathematical model of many physical phenomenon such as heat conduction in high polymer systems [3]. A lot of popular equations can be derived from Eq. (1.1). For instance when $\varphi(u) = a_1u + a_2u^2$, the Eq. (1.1) becomes Fisher equation and when $\varphi(u) = a_1u + a_3u^3$, the Eq. (1.1) becomes Newell-Whitchead equation [3, 4, 7, 8, 12].

In sequence we limit our consideration to derive a stable solution for the problem (1.1)-(1.3) with the presence of noise in the input data. A numerical marching scheme based on discrete mollification will be presented and the convergence and stability of this method will be proved.

For convenience we out line our procedure as follows:

In Section 2, we briefly review the mollification method. Section 3 contains our interest problem and its solution procedure based on mollification method and space marching scheme. In Section 4 the error analysis of the proposed numerical scheme is stated. Finally in Section 5 to support the previous section's results, some test problems are considered.

2 A brief review of mollification method

Let $\delta > 0$, $p > 0$ and $A_p = \left(\int_{-p}^{p} \exp(-s^2)ds\right)^{-1}$. The $\delta-$mollification of an integrable function is based on [9-13]

$$\rho_{\delta,p}(x) = \begin{cases} A_p\delta^{-1}\exp\left(-\frac{x^2}{\delta^2}\right), & |x| \le p\delta, \\ 0, & |x| > p\delta. \end{cases}$$

The $\delta-$mollifier $\rho_{\delta,p}$ is nonnegative $C^{\infty}(-p\delta, p\delta)$ function satisfying $\int_{-p}^{p} p_\delta(x)dx = 1$ and δ is called the radius of mollification. For notational purposes, we will denote the gaussian kernel by ρ_δ, dropping the dependence on the parameter p. We set $I = [0, 1]$ and $I_\delta = [p\delta, 1 - p\delta]$. Notice that the interval I_δ is nonempty whenever $p < 1/2\delta$. If f is a locally integrable function on I, we define its $\delta-$mollification on I by the convolution

$$J_\delta f(x) = (\rho_\delta * f)(x) = \int_{-\infty}^{\infty} \rho_\delta(x - s)f(x)ds = \int_{x-p\delta}^{x+p\delta} \rho_\delta(x - s)f(x)ds.$$

The $\delta-$mollification satisfies interesting consistency and stability estimates. Some of these results are listed as follows. The proofs of the statements in this section can be found in [10].

Theorem 2.1. 1. If $f(x)$ is uniformly Lipschitz on I, then there exists a constant C, independent of δ, such that

$$\| J_\delta f - f \|_{\infty, I_\delta} \le C\delta. \tag{2.1}$$

2. If $f(x)$ and $f^{\varepsilon}(x)$ are locally integrable functions on I and $\| f(x) - f^{\varepsilon}(x) \|_{\infty,I} \leq \varepsilon$, then there exists a constant C, independent of δ, such that

$$\begin{aligned} \| J_\delta f - J_\delta f^{\varepsilon} \|_{\infty,I_\delta} &\leq \varepsilon, &(2.2)\\ \| (J_\delta f)' - (J_\delta f^{\varepsilon})' \|_{\infty,I_\delta} &\leq C\frac{\varepsilon}{\delta}. &(2.3)\end{aligned}$$

3. If $f(x)$ and $f^{\varepsilon}(x)$ are locally integrable and uniformly Lipschitz on I with $\| f(x) - f^{\varepsilon}(x) \|_{\infty} \leq \varepsilon$, then

$$\| J_\delta f^{\varepsilon} - f \|_{\infty,I_\delta} \leq C\delta + \varepsilon. \tag{2.4}$$

Moreover, if f' is uniformly Lipschitz on I, then

$$\| (J_\delta f)' - f' \|_{\infty,I_\delta} \leq C\left(\delta + \frac{\varepsilon}{\delta}\right). \tag{2.5}$$

2.1 Discrete mollification

Suppose $K = \{x_j : j \in Z, 1 \leq j \leq M\} \subset I$, $x_{j+1} - x_j > d > 0, j \in Z$ and $0 \leq x_1 < x_2 < \cdots < x_M \leq 1$, where Z is the set of integers and d is a positive constant. Let $G = \{g_j\}_{j\in Z}$ be a discrete function defined on K and let $s_j = (1/2)(x_j + x_{j+1}), j \in Z$. The discrete $\delta-$mollification of G may define by

$$J_\delta G = \sum_{j=1}^{M}\left(\int_{s_{j-1}}^{s_j} \rho_\delta(x-s)ds\right) g_j.$$

Notice that $\sum_{j=1}^{M}(\int_{s_{j-1}}^{s_j} \rho_\delta(x-s)ds) = \int_{-p\delta}^{p\delta} \rho_\delta(s)ds = 1$. Let $\Delta x = \sup_{j\in Z}(x_{j+1} - x_j)$. Some results of the consistency, stability, and convergence of discrete δ-mollification are listed as follows.

Theorem 2.2. 1. If $g(x)$ is uniformly Lipschitz in I and $G = \{g_j = g(x_j) : j \in Z\}$ is the discrete version of g, then there exists a constant C independent of δ, such that

$$\| J_\delta G - g \|_{\infty,I_\delta} \leq C(\delta + \Delta x). \tag{2.6}$$

Moreover, if $g'(x) \in C^0(I)$ and $g''(x) \in C^0(I)$ then

$$\| (J_\delta G)' - g' \|_{\infty,I_\delta} \leq C\left(\delta + \frac{\Delta x}{\delta}\right). \tag{2.7}$$

2. If the discrete functions $G = \{g_j : j \in Z\}$ and $G^{\varepsilon} = \{g_j^{\varepsilon} : j \in Z\}$, which are defined on I, satisfy $\| G - G^{\varepsilon} \|_{\infty,I_\delta} \leq \varepsilon$, then we have

$$\begin{aligned} \| J_\delta G - J_\delta G^{\varepsilon} \|_{\infty,I_\delta} &\leq \varepsilon, &(2.8)\\ \| (J_\delta G)' - (J_\delta G^{\varepsilon})' \|_{\infty,I_\delta} &\leq \frac{C\varepsilon}{\delta}. &(2.9)\end{aligned}$$

3. If $g(x)$ is uniformly Lipschitz on I, let $G = \{g_j = g(x_j) : j \in Z\}$ be the discrete version of g and $G^\varepsilon = \{g_j^\varepsilon : j \in Z\}$ be the perturbed discrete version of g satisfying $\| G - G^\varepsilon \|_{\infty, I_\delta} \leq \varepsilon$. then,

$$\begin{aligned} \| J_\delta G - J_\delta g \|_{\infty, I_\delta} &\leq C(\delta + \Delta x), & (2.10)\\ \| J_\delta G - g \|_{\infty, I_\delta} &\leq C(\delta + \varepsilon + \Delta x). & (2.11)\end{aligned}$$

Moreover, if $g'(x) \in C^0(I)$ then

$$\begin{aligned} \| (J_\delta G^\varepsilon)' - (J_\delta g)' \|_{\infty, I_\delta} &\leq \frac{C}{\delta}(\varepsilon + \Delta x), & (2.12)\\ \| (J_\delta G^\varepsilon)' - g' \|_{\infty, I_\delta} &\leq C\left(\delta + \frac{\varepsilon}{\delta} + \frac{\Delta x}{\delta}\right). & (2.13)\end{aligned}$$

2.2 Numerical differentiation

Numerical differentiation is an ill-posed problem in the sense that small errors in the data might induce large errors in the computed derivative. The method that we present here uses the mollification method and allows for the stable reconstruction of the derivative of a function which is known approximately at a discrete set of data points. Let $G^\varepsilon = \{g_j^\varepsilon : j \in Z\}$ be the perturbed discrete data for the function g. In order, to recover the derivative g' from discrete noisy data, instead of utilizing $(d/dx)\rho_\delta$ and convolution with the data, computations are performed with a centered difference approximation of the mollified derivative $(d/dx)J_\delta G^\varepsilon$. Denote the centered difference operator by $\mathbf{D}$, i.e.,

$$\mathbf{D}f(x) = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}.$$

Following statements show some results regards to the stability and convergence of mollified derivative.

Theorem 2.3. If $g' \in C^1(R^1)$ and $G = \{g_j = g(x_j) : j \in Z\}$ is the discrete version of g, with G, G^ε satisfying $\| G - G^\varepsilon \|_{\infty, K} \leq \varepsilon$, then,

$$\| \mathbf{D}J_\delta G^\varepsilon - (J_\delta g)' \|_\infty \leq \frac{C}{\delta}(\varepsilon + \Delta x) + C_\delta(\Delta x)^2, \quad (2.14)$$

$$\| \mathbf{D}J_\delta G^\varepsilon - g' \|_\infty \leq C\left(\delta + \frac{\varepsilon}{\delta} + \frac{\Delta x}{\delta}\right) + C_\delta(\Delta x)^2. \quad (2.15)$$

Theorem 2.4. Suppose $G = \{g_j : j \in Z\}$ is a discrete function defined on a given set K and $\mathbf{D}_0^\delta(G) = \mathbf{D}(J_\delta G)(x) \mid_K$, then there exist a bound for this operator as

$$\| \mathbf{D}_0^\delta(G) \|_{\infty, K} \leq \frac{C}{\delta} \| G \|_{\infty, K} . \quad (2.16)$$

3 Regularized problem and marching scheme

In this section we will introduce a numerical marching scheme based on mollification method to find the solution of the problem (1.1)-(1.3) under the assumption that $\alpha(t)$ and $\beta(t)$ are only known approximately as $\alpha^\epsilon(t)$ and $\beta^\epsilon(t)$ such that $\|\alpha(t)-\alpha^\epsilon(t)\|_\infty \le \epsilon$ and $\|\beta(t)-\beta^\epsilon(t)\|_\infty \le \epsilon$. Because of the presence of the noise in the problem's data, we first stablize the problem using the mollification method. The regularized problem is formulated as follows. Determine $v(x,t), v_x(x,t) \in [0,1]\times[0.1]$ satisfying

$$v_t(x,t) = v_{xx}(x,t) + \varphi(v(x,t)), \qquad 0 \le x < 1, t > 0, \tag{3.1}$$

$$v(0,t) = J_{\delta_0}\alpha(t), \qquad t > 0, \tag{3.2}$$

$$v_x(0,t) = J_{\delta_0^*}\beta(t), \qquad t > 0, \tag{3.3}$$

where all δ-mollification are taken with respect to t and the radii of mollification, δ_0, δ_0^* are chosen automatically using the GCV method [1, 10, 11].

Now let M and N be positive integers, $h = \Delta x = 1/M$ and $k = \Delta t = 1/N$ be the parameters of the finite differences discretization of $I = [0,1]$. We introduce the following discrete functions
R_i^n: the discrete computed approximations of $v(ih, nk)$,
W_i^n: the discrete computed approximations of $v_t(ih, nk)$,
Q_i^n: the discrete computed approximations of $v_x(ih, nk)$.

The algorithm of space marching scheme may be written as follows

1. Select δ_0, δ_0^*.
2. Perform mollification of α^ε, β^ε in the interval $[0,1]$. Set
$$R_0^n = J_{\delta_0}\alpha^\varepsilon(nk), \quad Q_0^n = J_{\delta_0^*}\beta^\varepsilon(nk).$$
3. Perform mollified differentiation in time of $J_{\delta_0}\alpha^\varepsilon(nk)$. Set
$$W_0^n = \mathbf{D}_t(J_{\delta_0}\alpha^\varepsilon(nk)).$$
4. Initialize $i = 0$. Do while $i \le M-1$,
$$R_{i+1}^n = R_i^n + hQ_i^n, \tag{3.4}$$
$$Q_{i+1}^n = Q_i^n + h(W_i^n - \varphi(R_i^n)), \tag{3.5}$$
$$W_{i+1}^n = W_i^n + h\mathbf{D}_t(J_{\delta_i^*}Q_i^n). \tag{3.6}$$

From now on, if X_i^n is a discrete function, we denote $|X_i| = \max_n |X_i^n|$. We also consider a smoothing assumption to discuss the stability and convergence of the scheme as follows

$$u(x,t) \in C^2(I \times I).$$

4 Stability and convergence analysis

In this section, we analyze the stability and convergence of the proposed marching scheme.

Theorem 4.1 (Stability of the Algorithm)**.** There exists a constant C, such that

$$\max\{|R_M|, |Q_M|, |W_M|\} \leq C \max\{|Q_0|, |W_0|, |R_0|^m, |R_0|^{m-1}, \ldots, |R_0|, 1\} \tag{4.1}$$

Proof. Let $|\delta|_{-\infty} = \min_i(\delta_i, \delta_i^*)$. Applying Theorem 2.4 yields

$$|\mathbf{D}_t(Q_i^n)| \leq \frac{C}{|\delta|_{-\infty}} |Q_i^n|. \tag{4.2}$$

Now by using (3.6) and (4.2) we also have

$$|W_{i+1}^n| \leq |W_i^n| + h\frac{C}{|\delta|_{-\infty}} |Q_i^n| \leq (1 + hC_1) \max\{|W_i^n|, |Q_i^n|\}, \tag{4.3}$$

where $C_1 = \frac{C}{|\delta|_{-\infty}}$. Similarly using (3.4) and (3.5) yields

$$\begin{aligned} |R_{i+1}^n| &\leq |R_i^n| + h|Q_i^n| \leq (1+h) \max\{|R_i^n|, |Q_i^n|\}, & (4.4) \\ |Q_{i+1}^n| &\leq |Q_i^n| + h(|W_i^n| + |a_m||(R_i^n)^m| + |a_{m-1}||(R_i^n)^{m-1}| + \cdots + |a_1||R_i^n| + |a_0|) \\ &\leq (1 + (1 + |a_m| + |a_{m-1}| + \cdots + |a_0|)h) \max\{|Q_i^n|, |W_i^n|, |R_i^n|^m, |R_i^n|^{m-1}, \cdots, 1\}. \\ & & (4.5) \end{aligned}$$

Suppose

$$C_\delta = \max\{1 + |a_m| + |a_{m-1}| + \cdots + |a_1| + |a_0|, C_1\}$$

from (4.3)-(4.5), we obtain

$$\max\{|R_{i+1}|, |Q_{i+1}|, |W_{i+1}|\} \leq (1 + C_\delta) \max\{|Q_i|, |W_i|, |R_i^m, |R_i|^{m-1}, \ldots, |R_i|, 1\},$$

and iterating this last inequality M times, we have

$$\max\{|R_{i+1}|, |Q_{i+1}|, |W_{i+1}|\} \leq (1 + C_\delta)^M \max\{|Q_0|, |W_0|, |R_0^m, |R_0|^{m-1}, \ldots, |R_0|, 1\},$$

which implies

$$\max\{|R_M|, |Q_M|, |W_M|\} \leq (1 + C_\delta)(\exp(C_\delta)) \max\{|Q_0|, |W_0|, |R_0^m, |R_0|^{m-1}, \ldots, |R_0|, 1\}. \tag{4.6}$$

This complete the proof of this theorem. □

Theorem 4.2 (Formal convergence)**.** For fixed δ as h, k and ε tend to zero, the discrete mollified solution converges to the mollified exact solution restricted to the grid points.

Proof. From the definitions of discrete error functions let

$$\Delta R_i^n = R_i^n - v(ih, nk),\ \Delta Q_i^n = Q_i^n - v_x(ih, nk),\ \Delta W_i^n = W_i^n - v_t(ih, nk).$$

Using Taylor series, we obtain some useful equations satisfied by the mollified solution v, namely,

$$\begin{aligned} v((i+1)h, nk) &= v(ih, nk) + hv_x(ih, nk) + O(h^2), && (4.7)\\ v_x((i+1)h, nk) &= v_x(ih, nk) + h(v_t(ih, nk) - \varphi(v(ih, nk))) + O(h^2), && (4.8)\\ v_t((i+1)h, nk) &= v_t(ih, nk) + h\frac{d}{dt}v_x(ih, nk) + O(h^2). && (4.9) \end{aligned}$$

On the other hand, one may write

$$\begin{aligned} \Delta R_{i+1}^n &= \Delta R_i^n + (R_{i+1}^n - R_i^n) - (v((i+1)h, nk) - v(ih, nk))\\ &= \Delta R_i^n + hQ_i^n - hv_x(ih, nk) + O(h^2)\\ &= \Delta R_i^n + h\Delta Q_i^n + O(h^2). \end{aligned} \tag{4.10}$$

$$\begin{aligned} \Delta Q_{i+1}^n &= \Delta Q_i^n + (Q_{i+1}^n - Q_i^n) - (v_x((i+1)h, nk) - v_x(ih, nk))\\ &= \Delta Q_i^n + h((W_i^n - \varphi(v(ih, nk))) - h(v_t(ih, nk) - \varphi(v(ih, nk))) + O(h^2)\\ &= \Delta Q_i^n + h\Delta W_i^n + O(h^2). \end{aligned} \tag{4.11}$$

$$\begin{aligned} \Delta W_{i+1}^n &= \Delta W_i^n + (W_{i+1}^n - W_i^n) - (v_t((i+1)h, nk) - v_t(ih, nk))\\ &= \Delta W_i^n + h\mathbf{D}_t(J_{\delta_i^*}Q_i^n) - hv_{tx}(ih, nk) + O(h^2)\\ &= \Delta W_i^n + h(\mathbf{D}_t(J_{\delta_i^*}Q_i^n) - v_{tx}(ih, nk)) + O(h^2). \end{aligned} \tag{4.12}$$

Now from equalities (4.10)-(4.12),using the error estimates of discrete mollification from theorem 2.3 we have

$$\begin{aligned} |\Delta R_{i+1}^n| &\le |\Delta R_i^n| + h|\Delta Q_i^n| + O(h^2),\\ |\Delta W_{i+1}^n| &\le |\Delta W_i^n| + h|\mathbf{D}_t(J_{\delta_i^*}Q_i^n) - v_{tx}(ih, nk)| + O(h^2),\\ &\le |\Delta W_{i,n}| + h\left(C\frac{|\Delta Q_i^n| + k}{|\delta|_{-\infty}} + C_{\delta^*}k^2\right) + O(h^2).\\ |\Delta Q_i^n| &\le |\Delta Q_i^n| + h|\Delta W_i^n| + O(h^2). \end{aligned}$$

Suppose

$$\Delta_i = \max\{|\Delta R_i^n|, |\Delta W_i^n|, |\Delta Q_i^n|\},\ C_0 = \max\left\{1, \frac{C}{|\delta|_{-\infty}}\right\}, C_1 = \max\left\{\frac{ck}{|\delta|_{-\infty}} + C_{\delta^*}k^2\right\}$$

Then we obtain

$$\begin{aligned} \Delta_{i+1} &\le (1 + hC_0)\Delta_i + hC_1 + O(h^2)\\ &\le (1 + hC_0)(\Delta_i + C_1) + O(h^2), \end{aligned} \tag{4.13}$$

and after L iterations

$$\Delta_L \leq \exp(C_0)(\Delta_0 + C_1). \tag{4.14}$$

Moreover from

$$\begin{aligned}
|\Delta R_0^n| &= |R_0^n - v(0,nk)| = |J_{\delta_0}\alpha^\varepsilon(nk) - v(0,nk)| \leq C(\varepsilon + k), \\
|\Delta Q_0^n| &= |Q_0^n - v_x(0,nk)| = |J_{\delta_0^*}\beta^\varepsilon(nk) - v(0,nk)| \leq C(\varepsilon + k), \\
|\Delta W_0^n| &= |\mathbf{D}_t(J_{\delta_0}\alpha^\varepsilon(nk)) - v_t(0,nk)| \leq \frac{C}{\delta_0}(\varepsilon + k) + C_\delta K^2,
\end{aligned}$$

we see that when ε, h, and k tend to 0, Δ_0 and C_1 tend to 0. Consequently $(\Delta_0 + C_1)$ tends to 0 and so does Δ_L and this complete the proof of this theorem. □

5 Numerical examples

In this section, we present some numerical results of interest. In all cases, without loss of generality, we set $p = 3$. These values are appropriate because the difference between ρ_δ's when $p = 3$ and $p > 3$ is insignificant. The radii of mollification are always chosen automatically using the mollification and GCV methods.
Discretized measured approximations of boundary data are modeled by adding random errors to the exact data functions. For example, for the boundary data function $h(x,t)$, its discrete noisy version is generated by

$$h_{j,n}^\varepsilon = h(x_j, t_n) + \varepsilon_{j,n}, \qquad j = 0, 1, \ldots, N, n = 0, 1, \ldots, T,$$

where the$(\varepsilon_{j,n})$'s are Gaussian random variables with variance ε^2.
The errors exact and approximate solution are measured by the relative weighted l_2-norm given by

$$\frac{\left[(1/(M+1)(N+1))\Sigma_{i=0}^M \ \Sigma_{j=0}^N (v(ih,jl) - R_i^j)^2\right]^{1/2}}{\left[(1/(M+1)(N+1))\Sigma_{i=0}^M \ \Sigma_{j=0}^N (v(ih,jl))^2\right]^{1/2}}.$$

Example 5.1. As the first example, consider following nonlinear Cauchy problem

$$\begin{aligned}
u_t &= u_{xx} + u(1-u)(4-u), && 0 < x < 1, \ 0 < t < 1, \\
u(0,t) &= 2 + 2\tanh(2t+1), && 0 \leq t \leq 1, \\
u_x(0,t) &= 2\sqrt{2}(1 - (\tanh^2(2t+1)), && 0 \leq t \leq 1.
\end{aligned}$$

The exact solution for $u(x,t)$ may be derived as

$$u(x,t) = 2 + 2\tanh(\sqrt{2}x + 2t + 1).$$

The figure 1 and table 1 show the comparison between exact and numerical solutions and the relative l_2 errors.

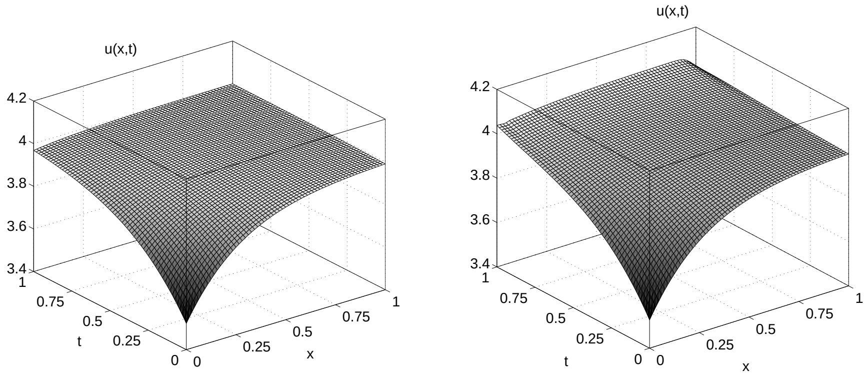

Figure 1: Exact (right side) and numerical (left side) solutions.

Table 1: Relative l_2 error norms for example 5.1

M	N	ε	u	u_t	u_x
64	64	0.001	0.0072928	0.14286	0.25619
64	128	0.001	0.0072113	0.181	0.25686
64	256	0.001	0.007202	0.25634	0.2616
64	64	0.010	0.064593	0.28813	0.3452
64	128	0.010	0.06453	0.31147	0.3571
64	256	0.010	0.064523	0.32158	0.36106

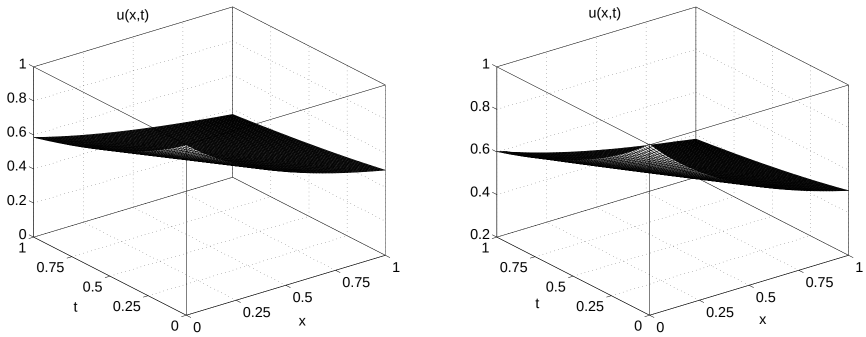

Figure 2: Exact (right side) and numerical (left side) solutions.

Table 2: Relative l_2 error norms for example 5.2

M	N	ε	u	u_t	u_x
64	64	0.001	0.014265	0.018557	0.050753
64	128	0.001	0.014332	0.013466	0.051521
64	256	0.001	0.014388	0.019122	0.052022
64	64	0.010	0.13033	0.078138	0.41362
64	128	0.010	0.13042	0.080611	0.41483
64	256	0.010	0.13048	0.081887	0.41545

Example 5.2. Consider following problem

$$\begin{aligned} u_t &= u_{xx} - u^2 - u^3, && 0 < x < 1,\ 0 < t < 1, \\ u(0,t) &= \frac{1}{t+1}, && 0 \le t \le 1, \\ u_x(0,t) &= \frac{-\sqrt{2}}{2(t+1)^2}, && 0 \le t \le 1. \end{aligned}$$

The exact solution of this problem is

$$u(x,t) = \frac{1}{t + \frac{\sqrt{2}}{2}x + 1}.$$

The figure 2 and table 2 show the comparison between exact and numerical solutions and the relative l_2 errors.

Example 5.3. As the final test problem, consider following problem

$$\begin{aligned} u_t &= u_{xx} + 6u(1-u), && 0 < x < 1,\ 0 < t < 1, \\ u(0,t) &= \frac{1}{(e^{x-5t}+1)^2}, && 0 \le t \le 1, \\ u_x(0,t) &= \frac{-2e^{x-5t}}{(e^{x-5t}+1)^3}, && 0 \le t \le 1. \end{aligned}$$

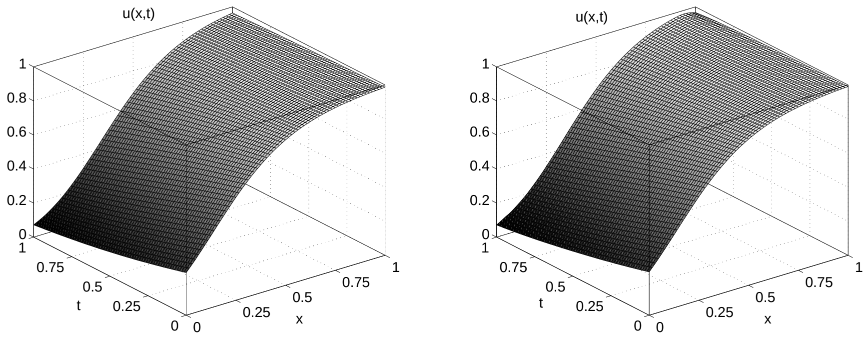

Figure 3: Exact (right side) and numerical (left side) solutions.

Table 3: Relative l_2 error norms for example 5.3

M	N	ε	u	u_t	u_x
64	64	0.001	0.014804	0.043989	0.11316
64	128	0.001	0.014875	0.050383	0.11435
64	256	0.001	0.014907	0.071938	0.11853
64	64	0.010	0.13054	0.13202	1.0878
64	128	0.010	0.13063	0.13381	1.0863
64	256	0.010	0.13062	0.14599	1.0841

One may find the exact solution as

$$u(x,t) = \frac{1}{(1+e^{x-5t})^2}.$$

The figure 3 and table 3 show the comparison between exact and numerical solutions and the relative l_2 errors.

6 Conclusion

In this work, a class of semi-linear Cauchy problems is investigated. The noisy boundary conditions are considered and a spatial regularization method based on mollification scheme and space marching method is applied to solve the proposed non-well posed problem. The error analysis in this study shows the stability and convergence of the proposed method.

References

[1] B. Anderssen, F. de Hogg, M. Hegland. A stable finite difference ansatz for higher order differentiation of nonexact data. Bull. Austral. Math. Soc., 1998, 58: 223 - 232, doi:

10.1017/S0004972700032196.

[2] C. Atkinson, G.E.H. Reuter, C.J. Ridler-Rowe. Travelling wave solutions for some nonlinear diffusion equation. SIAM J. Math. Anal., 1981,12: 880 - 892.

[3] G. Bluman, S. Kumei. On the remarable nonlinear diffusion equation. J. Math. Phys., 1980, 5: 1019 - 1023, doi: 10.1063/1.524550.

[4] P. Brazhnik, J. Tyson. On traveling wave solutions of Fisher's equation in two spatial dimensions. SIAM J. Appl. Math., 1999, 60: 371 - 391.

[5] C. Coles, D.A. Murio. Simultaneous space diffusivity and source term reconstruction in 2D IHCP. Comput. Math. Applic., 2001, 12: 1549 - 1564.

[6] D.S. Jone, B.D. Sleeman. Differential Equations and Mathematical Biology. Chapman & Hall/CRC, New York, 2003.

[7] T. Kawahara, M. Tanaka. Interactions of traveling fronts: an exact solution of a nonlinear diffusion equation. Phy. Lett. A, 1983, 97: 311 - 314, doi: 10.1016/0375-9601(83)90648-5.

[8] W. Malfliet. Solitary wave solutions of nonlinear wave equations. American J. Phy., 1992, 60: 650 - 654, doi: 10.1119/1.17120.

[9] D.A. Murio, ZH. Yi. Source Term Identification in 1-D IHCP. Comput. Math. Applic., 2004, 47: 1921 - 1933, doi: 10.1016/j.camwa.2002.11.025.

[10] D.A. Murio. Stable numerical solution of a fractional-diffusion inverse heat conduction problem. Comput. Math. Applic., 2007, 53: 1492 - 1501, doi: 10.1016/j.camwa.2006.05.027.

[11] D.A. Murio. The Mollification Method and the Numerical Solution of Ill-Posed Problems. John Wiley and Sons, New York, 1993, doi: 10.1002/9781118033210.

[12] J. Sheratt. On the transition from initial data traveling waves in the Fisher-KPP equation. Dyn. Stab. Sys., 1998, 13: 167 - 174.

[13] X.Y. Wang. Exact and explicit solitary wave solutions for the generalized Fisher equation. Phy. Lett. A, 1988, 131: 277 - 279, doi: 10.1016/0375-9601(88)90027-8.